

Dynamics of Dollard asymptotic variables. Asymptotic fields in Coulomb scattering

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Abstract

Generalizing Dollard's strategy, we investigate the structure of the scattering theory associated to any large time reference dynamics $U_D(t)$ allowing for the existence of Møller operators. We show that (for each scattering channel) $U_D(t)$ uniquely identifies, for $t \rightarrow \pm\infty$, *asymptotic dynamics* $U_{\pm}(t)$; they are unitary *groups* acting on the scattering spaces, satisfy the Møller interpolation formulas and are interpolated by the S -matrix. In view of the application to field theory models, we extend the result to the adiabatic procedure. In the Heisenberg picture, asymptotic variables are obtained as LSZ-like limits of Heisenberg variables; their time evolution is induced by $U_{\pm}(t)$, which replace the usual free asymptotic dynamics. On the asymptotic states, (for each channel) the Hamiltonian can be written in terms of the asymptotic variables as $H = H_{\pm}(q_{out/in}, p_{out/in})$, $H_{\pm}(q, p)$ the generator of the asymptotic dynamics. As an application, we obtain the asymptotic fields $\psi_{out/in}$ in repulsive Coulomb scattering by an LSZ modified formula; in this case, $U_{\pm}(t) = U_0(t)$, so that $\psi_{out/in}$ are *free* canonical fields and $H = H_0(\psi_{out/in})$.

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1 Introduction

As clarified by Dollard [5] [6], long range interactions require a substantial modification of the standard scattering theory and for potential scattering the situation is well understood [3].

The possibility of exploiting Dollard's strategy, i.e., the use of a large time reference dynamics which is not free and not even a group, for controlling the infrared problem in QED is an interesting and, in our opinion, open problem. Quite generally, one may ask what are the indications of Dollard strategy for a Lehmann-Symanzik-Zimmermann (LSZ) condition for the construction of asymptotic charged fields.

An extension of Dollard approach to QED has been proposed by Kulish and Faddeev [11] and by Rohrlich [9], involving both the identification of a Dollard reference dynamics for large times, which takes into account the Coulomb distortion and the infinite photon emission, and the identification of the asymptotic fields with those proposed by Zwanziger and arising from generalized LSZ limits [19].

The outcome of that proposal represent a substantial departure from the standard framework; in fact, the time evolution of the asymptotic fields is not a group and the S -matrix is not invariant under it.

In Ref. [9], such features arise as a consequence of the identification of the dynamics of the asymptotic fields with the Dollard modification of the free dynamics; more generally, they stem from the incorporation of Coulomb and soft photons effects in the dynamics of asymptotic fields [19]. It has also been argued that "distorted" asymptotic fields, with a dynamics which is not a group, characterize the field theory version of Coulomb scattering [17]. This implies a substantial change from the LSZ and Haag-Ruelle (HR) notion of asymptotic fields.

The aim of this note is to investigate *the general structure of the scattering theory associated to a large time reference dynamics $U_D(t, s)$, satisfying the flow property (but not the group property), which allows for the existence of Møller operators as strong limits.*

This will be done in a framework admitting different scattering channels, both in the Schroedinger and Heisenberg pictures and also in the presence of an adiabatic switching.

The main result of Section 2 is that any reference dynamics leading to the existence of Møller operators Ω_{\pm} as strong limits for $t \rightarrow \pm\infty$

(with respect to a given dynamics $U(t)$) uniquely determines an *asymptotic dynamics* $U_{\pm}(t)$, which is a *strongly continuous unitary group* and *satisfies the standard interpolation formula*,

$$U(t)\Omega_{\pm} = \Omega_{\pm}U_{\pm}(t) . \quad (1.1)$$

Even if $U_D(t, s)$ is asymptotic to a free dynamics U_0 for large $|t|$, namely $U_D(t + \tau, s) \sim U_0(\tau) U_D(t, s)$, $U_{\pm}(t)$ needs not to be free.

The implications on the construction of Heisenberg asymptotic variables and S -matrix are discussed in Section 3. The Heisenberg asymptotic variables, conventionally $q_{out/in}(t), p_{out/in}(t)$, are covariant under $U(t)$ and, on the scattering space, the Hamiltonian can be written as

$$H = H_{out/in}(q_{out/in}, p_{out/in}) , \quad (1.2)$$

where $H_{out/in}$ is the generator of $U_{\pm}(t)$. Thus, one recovers all the standard properties, with the only replacement of the free Hamiltonian H_0 with $H_{out/in}$.

For of the extension of Dollard approach to the construction of Møller operators in quantum field theory (QFT) models it is convenient to introduce an *adiabatic regularization* and one must control the effects arising in its removal (Section 4). Quite generally, with a proper treatment of persistent effects like mass renormalization counterterms, one obtains as above an asymptotic dynamics which is a group and satisfies eq. (1.1) (Section 4).

In Section 5, the analysis of Sections 2, 3 is applied to the control of asymptotic fields for (repulsive) Coulomb interactions; they are defined as strong asymptotic limits, are space-time covariant free fields (with no Coulomb distortion) and are interpolated by the Heisenberg S -matrix, which is space-time translation invariant. In this case, the adiabatic procedure of Section 4 displays the factorization of infrared divergences. Thus, the Coulomb interaction and the corresponding distortion in Dollard reference dynamics do not imply a distortion [17] of the asymptotic dynamics, which remains free.

An asymptotic dynamics which is still a group, in accord with the results of Sections 2 and 4, but is not free, arises in QFT models with infinite photon emission. In fact, the present framework and analysis apply to a model with realistic photon emission, describing classical particles with Coulomb interaction and translation invariant coupling to the quantized electromagnetic field [14].

2 Dollard asymptotic limits and asymptotic dynamics

The important contribution by Dollard on Coulomb scattering has played a crucial role in the discussion of scattering in the presence of long range interactions and of the infrared problem in quantum electrodynamics (QED) [5] [6] [18] [3] [11] [9].

The main idea is that for the definition of the asymptotic limit one has to replace the free dynamics by a “distorted one”, which is not a one-parameter group. In the interaction picture this amounts to replace $U_0(t) = e^{-iH_0 t}$ by $U_D(t)$, with

$$\frac{dU_D(t)}{dt} = -iH_D(t)U_D(t), \quad (2.1)$$

where $H_D(t)$ is time dependent; for Coulomb scattering

$$H_D(t) = H_0 + \frac{\alpha m}{|\mathbf{p}|t}, \quad U_D(t) = U_0(t) \exp\left(-i\frac{\alpha m}{|\mathbf{p}|} \text{sign } t \log |t|\right). \quad (2.2)$$

The role of such a distortion is to allow for the existence of the strong limits

$$\Omega_{\pm} \equiv s - \lim_{t \rightarrow \pm\infty} e^{iHt} U_{D\pm}(t), \quad (2.3)$$

which do not exist for $e^{iHt} U_0(t)$, and of the S-matrix $S = \Omega_+^* \Omega_-$.

It is worthwhile to stress that even if $U_D(t)$ is not a group, in the literature [17] [9] it has been identified with “the dynamics of the asymptotic states”, with the consequence that time covariance is inevitably lost. This is clearly a serious obstruction for the construction of space time covariant asymptotic fields and in general for a LSZ approach. In fact, Zwanziger proposal [19] for asymptotic fields in QED explicitly displays such a loss of covariance.

The aim of this Section is to re-discuss Dollard’s strategy and its implications in general, in order to question the above identification of the asymptotic dynamics and the loss of the group law.

To this purpose we shall start by analyzing the properties of the “large time reference dynamics” $U_{D\pm}(t, s)$ following merely by the fact that they allow for the existence of the Møller operators.

2.1 The group of the asymptotic dynamics

We consider the scattering problem for the time evolution $U(t) = e^{-iHt}$, in a Hilbert space \mathcal{H} , with respect to a “large time reference dynamics” defined by a family of unitary operators $U_D(t) = U_D(t, 0)$, such that, for all “scattering vectors” ψ ,

$$\|U(t)\psi - U_D(t)\varphi\| \rightarrow 0, \quad \text{for } t \rightarrow \pm\infty. \quad (2.4)$$

This means that for large $|t|$, $U(t)\psi$ is described by the reference dynamics $U_D(t)$ of the “reference vectors” φ . In order to cover the case of different scattering channels, indexed by α (e.g. corresponding to bound states), it is convenient to consider as reference vectors elements of *scattering spaces* \mathcal{H}_α^\pm , and reformulate the above convergence as the existence of

$$\Omega_\pm^\alpha = \lim_{t \rightarrow \pm\infty} U^*(t) J_\pm^\alpha U_{D\pm}^\alpha(t); \quad \Omega_\pm^\alpha : \mathcal{H}_\alpha^\pm \rightarrow \mathcal{H}, \quad (2.5)$$

where $J_\pm^\alpha : \mathcal{H}_\alpha^\pm \rightarrow \mathcal{H}$ are isometric operators describing different channels, and $U_{D\pm}^\alpha(t)$ are the reference dynamics in \mathcal{H}_α^\pm (see [3]).

Further requirements for the choice of the large time reference dynamics $U_{D\pm}^\alpha(t)$, in particular its relation with a free dynamics, will be discussed later.

Such a formulation substantially reproduces the formulation of the scattering in the interaction picture, with the generalization in which the reference dynamics needs not to be a group. In general, the states of $\mathcal{H}^\pm \equiv \oplus_\alpha \mathcal{H}_\alpha^\pm$ may be described in terms of “scattering” algebras \mathcal{A}^\pm of operators acting in \mathcal{H}^\pm , respectively, and $U_{D\pm}(t)$ may be identified with functions of such variables (i.e., elements of the von Neumann closure of \mathcal{A}^\pm).

The operators Ω_\pm^α are automatically isometric; $\mathcal{H}_{(\pm)}^\alpha \equiv \Omega_\pm^\alpha \mathcal{H}_\alpha^\pm$ are interpreted as the spaces of *asymptotic states* in \mathcal{H} for $t \rightarrow \pm\infty$ and will be assumed to be orthogonal and stable under time evolution

$$U(t) \Omega_\pm^\alpha \mathcal{H}_\alpha^\pm = \Omega_\pm^\alpha \mathcal{H}_\alpha^\pm. \quad (2.6)$$

Property (2.6) follows from eq. (2.3) in the standard scattering theory, where the reference dynamics $U_D(t)$ is a group. Its failure would lead to serious problems for the formulation of scattering theory (see below).

Until the end of Section 2.2, we shall work in a fixed channel and omit the index α .

The fulfillment of eqs. (2.5), (2.6) is unaffected if $U_{D\pm}(t)$ is replaced by $U_{D\pm}(t)V_{\pm}$, with V_{\pm} unitary operators; this amounts to replace Ω_{\pm} by $\Omega_{\pm}V_{\pm}$.

Proposition 2.1 *Assuming eq.(2.5),
i) the following weak limit exist, $\forall s \in \mathbf{R}$:*

$$U_{\pm}(s) \equiv w - \lim_{t \rightarrow \pm\infty} U_{D\pm}(t)^{-1} U_{D\pm}(t+s) = \Omega_{\pm}^* U(s) \Omega_{\pm}, \quad (2.7)$$

ii) the stability condition (2.6) is a necessary condition for any of the following properties:

- 1) $U_{\pm}(s)$ are isometric operators, equivalently, the limit in eq.(2.7) is strong;
- 2) $U_{\pm}(s)$, is a one-parameter group;
- 3) $U(s)\Omega_{\pm} = \Omega_{\pm}U_{\pm}(s)$.

Proof. In fact, $\forall \psi, \varphi \in \mathcal{H}^{\pm}$, one has

$$(\psi, U_D(t)^{-1} U_D(t+s) \varphi) = (U^*(t) J U_D(t) \psi, U(s) U^*(t+s) J U_D(t+s) \varphi)$$

$$\longrightarrow_{t \rightarrow \pm\infty} (\Omega_{\pm} \psi, U(s) \Omega_{\pm} \varphi) = (\psi, \Omega_{\pm}^* U(s) \Omega_{\pm} \varphi). \quad (2.8)$$

Moreover, $U_{\pm}(s)$ isometric implies

$$\mathbf{1} = U_{\pm}^*(s) U_{\pm}(s) = \Omega_{\pm}^* U(-s) \Omega_{\pm} \Omega_{\pm}^* U(s) \Omega_{\pm} \quad (2.9)$$

and by multiplication by Ω_{\pm}^* and Ω_{\pm} on the right and on the left, respectively, one has, $\forall \psi \in \Omega_{\pm} \Omega_{\pm}^* \mathcal{H} \equiv P_{\pm} \mathcal{H}$,

$$(U(s) \psi, P_{\pm} U(s) \psi) = (\psi, P_{\pm} \psi) = (\psi, \psi), \quad (2.10)$$

i.e., $P_{\pm} U(s) \psi = U(s) \psi$; therefore $U(s)$ leaves $P_{\pm} \mathcal{H}$ stable.

Similarly, if $U_{\pm}(s)$ is a one-parameter group, one has $U_{\pm}(s) U_{\pm}(-s) = \mathbf{1}$ and, since eq. (2.7) implies $U_{\pm}^*(s) = U_{\pm}(-s)$, $U_{\pm}(s)$ is unitary. Point 3) is immediate. \square

Quite generally, stability follows from any interpolation equation,

$$U(s)W = WV(s) \Rightarrow U(s)W\mathcal{H} \subset W\mathcal{H}; \quad (2.11)$$

however, given $U(t)$ and $W = \Omega_{\pm}$, $U_{\pm}(s)$ is the only possible solution of such an equation since $U(s)\Omega_{\pm} = \Omega_{\pm}V_{\pm}(s)$ implies $V_{\pm}(s) = U_{\pm}(s)$ by eq. (2.7).

Proposition 2.2 shows that stability under time evolution, eq. (2.6), implies that the limit (2.7) is strong; equivalently, that for $t \rightarrow \pm\infty$

$$U_{D\pm}(t+s) \sim U_{D\pm}(t) U_{\pm}(s), \quad (2.12)$$

with $U_{\pm}(s)$ a one-parameter group, *uniquely determined* by eq. (2.12). Here and in the following, \sim denotes strong convergence to 0 of the difference.

This means that for large $|t|$, the increments $t \rightarrow t+s$ are described by the action of $U_{\pm}(s)$ on the right, briefly $U_{D\pm}(t)$ has the one-parameter groups U_{\pm} as its (*unique*) *right asymptotic group*.

For simplicity, we shall consider the case $t \rightarrow \infty$, the extension to $t \rightarrow -\infty$ being straightforward; in most cases we shall omit the subscripts \pm , denote the corresponding *scattering spaces* \mathcal{H}^{\pm} by \mathcal{H}^{∞} and U_{+} by U_{as} .

Proposition 2.2 *Equations (2.5),(2.6) imply the existence of the following strong limit, defining a strongly continuous one-parameter group $U_{as}(s) = e^{-iH_{as}s}$ (the asymptotic dynamics)*

$$U_D(t)^{-1} U_D(t+s) \xrightarrow{s \rightarrow t \rightarrow \infty} U_{as}(s). \quad (2.13)$$

$U_{as}(s)$ is the unique solution of eq. (2.12), for $t \rightarrow \infty$, and satisfies the following **interpolation formula**

$$U(s)\Omega = \Omega U_{as}(s), \quad \text{on } \mathcal{H}^{\infty}. \quad (2.14)$$

Lemma 2.3 *If a sequence of isometric operators U_n converges strongly, for $n \rightarrow \infty$, $U_n \xrightarrow{s} \Omega$, then Ω is isometric and*

$$s - \lim_{n \rightarrow \infty} U_n^* \Omega = \mathbf{1} = \Omega^* \Omega, \quad (2.15)$$

i.e.

$$s - \lim_{n \rightarrow \infty} U_n^* = \Omega^*, \quad \text{on } \Omega \mathcal{H}. \quad (2.16)$$

Proof. In fact, $\forall \psi \in \mathcal{H}$, $U_n^* \Omega \psi = U_n^*(U_n \psi + \chi_n)$, with $\chi_n \xrightarrow{s}_{n \rightarrow \infty} 0$.
 \square

Proof (of Proposition 2.2). By eq. (2.5), $\forall \psi \in \mathcal{H}^\infty$

$$U_D(t)^{-1} U_D(t+s) \psi = U_D(t)^{-1} J^* U(t+s) U(-t-s) J U_D(t+s) \psi =$$

$$(U(t)^{-1} J U_D(t))^* U(s) (\Omega \psi + \chi(t)), \quad (2.17)$$

with $\chi(t) \xrightarrow{s} 0$, for $t \rightarrow \infty$.

By eq. (2.6) and Lemma 2.3, the right hand side converges to

$$\Omega^* U(s) \Omega \psi \equiv U_{as}(s) \psi, \quad (2.18)$$

and, since by eq. (2.15) $\Omega \Omega^* = \mathbf{1}$ on $U(s) \Omega \mathcal{H}^\infty \subseteq \Omega \mathcal{H}^\infty$,

$$U_{as}^*(s) U_{as}(s) = \Omega^* U(-s) \Omega \Omega^* U(s) \Omega = \mathbf{1};$$

furthermore, $U_{as}(s)^* = U_{as}(-s)$ gives $U_{as}(s) U_{as}(s)^* = \mathbf{1}$.

Strong continuity follows from the definition of $U_{as}(s)$, eq. (2.18); moreover, since $\Omega \Omega^* = \mathbf{1}$ on $\Omega \mathcal{H}^\infty$, eq. (2.6) implies the group law and eq. (2.14), the latter by multiplying eq. (2.18) by Ω . \square

Remark 1. Given eq. (2.5), by Prop. (2.1) strong convergence of the l.h.s. of eq. (2.13) is actually equivalent to eq. (2.6). Equation (2.14) shows that the time evolution of the asymptotic states in \mathcal{H} , i.e. of the vectors in $\Omega \mathcal{H}^\infty$, is given by a *one-parameter group* $U_{as}(t)$, uniquely determined by $U_D(t)$. This property, which holds for Coulomb scattering with $H_{as} = H_0$ [6], is therefore much more general, only depending on the existence of the asymptotic limits, eq. (2.5), and on the stability of the scattering states under the time evolution, eq. (2.6).

Remark 2. A free dynamics $U_0(t)$ has not entered in the above analysis and in particular $U_{as}(t)$ needs not to be a free dynamics. On the other side, the relevance of $U_{as}(t)$ is clearly displayed by eq. (2.14), which gives it uniquely in terms of the Møller operators. Moreover, if $V(t)$ is a family of unitary operators, any interpolation formula

$$U(t)W = WV(t), \quad \text{with } W^*W = \mathbf{1}, \quad \text{on } \mathcal{H}^\infty \quad (2.19)$$

implies that $W\mathcal{H}^\infty$ is stable under $U(t)$ and $V(t)$ is a one-parameter group. Therefore, $U_D(t)$ cannot be a candidate for such an interpolation

with respect to $U(t)$, (contrary to statements appeared in the literature [17]), unless it has the group property.

Remark 3. The above results are relevant for the construction of asymptotic limits in models with soft photon emission. In fact, as a consequence of the absence of charged one-particle states with a definite energy dispersion law $E(\mathbf{p})$ (as implied by the presence of an infinite number of asymptotic photons), the spectrum of H on the scattering states cannot be that of the free Hamiltonian H_0 and therefore eq. (2.14) cannot hold with $H_{as} = H_0$. This is explicitly shown by the model of Ref. [14].

Remark 4. It is worthwhile to remark that $\tilde{U}_D(t) \equiv U_D(t)V$ gives

$$\tilde{U}_{as}(t) = V^* U_{as}(t) V, \quad \tilde{\Omega} = \Omega V. \quad (2.20)$$

This occurs if a change of initial time $U(t) \rightarrow U(t-s)$ is accompanied by $U_D(t) \rightarrow U_D(t)U_D(s)^* = U_D(t, s)$ (see Appendix A) rather than by $U_D(t) \rightarrow U_D(t-s)$. In this case, the asymptotic dynamics, the corresponding Møller operators and the S -matrix are given by

$$U_D(s) U_{as}(t) U_D(s)^*, \quad \Omega(s) = U(s)\Omega(0)U_D(s)^*,$$

$$S(s) = \Omega_+^*(s)\Omega_-(s) = U_D(s)S(0)U_D(s)^*. \quad (2.21)$$

2.2 Asymptotically free reference dynamics

Up to now, $U_D(t)$ has been only constrained to satisfy eqs. (2.5), (2.6). In many cases, however, $U_D(t)$ is constructed as a modification of the free dynamics $U_0(t) \equiv e^{-iH_0 t}$, needed when $U(t)^{-1}JU_0(t)$ does not converge for $t \rightarrow \infty$. If the corrections to the free Hamiltonian are small for large times, in a suitable sense, interesting relations arise between $U_D(t)$ and $U_0(t)$. In Appendix A, a notion is given of “asymptotically vanishing” modification of the free Hamiltonian, which implies (Proposition A.3) the following relation between $U_D(t)$ and $U_0(t)$.

Definition 2.4 $U_D(t)$ is said to be **asymptotically free** if it has U_0 as a left asymptotic group,

$$U_D(t+s) \sim_{|t| \rightarrow \infty} U_0(s) U_D(t), \quad (2.22)$$

uniformly for s in finite intervals.

Eq. (2.22) means that, for large $|t|$, the increments $t \rightarrow t + s$ are described by the action of $U_0(s)$ *on the left*. Contrary to the case of a right asymptotic group (eq. (2.12), eq. (2.22) does not identify a unique left asymptotic group; in particular, it is always satisfied by $U(t)$, as a consequence of eq. (2.5). Therefore, eq. (2.22) should not be viewed as a reconstruction of U_0 from $U_D(t)$, but rather as a *constraint on $U_D(t)$, given U_0* .

The most important consequence of the above notion is the following.

Proposition 2.5 *If $U_D(t)$ is asymptotically free, then*

$$U_{as}(s) = s - \lim_{t \rightarrow \infty} U_D(t)^{-1} U_0(s) U_D(t), \quad (2.23)$$

and $\sigma(H_{as}) \subseteq \sigma(H_0)$.

Proof. Eq. (2.23) immediately follows from eq. (2.13) and eq. (2.22). Uniform convergence for s in finite intervals implies convergence in the sense of tempered distributions.

Moreover, $\forall f \in \mathcal{S}(\mathbf{R})$, denoting by \tilde{f} its Fourier transform,

$$\begin{aligned} U_{as}(f) &= \int_{\sigma(H_{as})} dE_{as}(\lambda) e^{-i\lambda s} f(s) ds = \\ &= \int_{\sigma(H_{as})} dE_{as}(\lambda) \tilde{f}(\lambda) = \lim_{t \rightarrow \infty} U_D(t)^{-1} \int_{\sigma(H_0)} dE_0(\lambda) \tilde{f}(\lambda) U_D(t), \end{aligned}$$

and, therefore, the integral vanishes if $\text{supp } \tilde{f} \cap \sigma(H_0) = \emptyset$, which implies $\sigma(H_{as}) \subseteq \sigma(H_0)$. \square

Remark Even if $U(t)^{-1} J U_0(t)$ does not converge, $U(t)$ may satisfy

$$U_0(t+s)^{-1} J^* U(t+s) \sim U_0(t)^{-1} J^* U(t), \quad \text{on } \Omega \mathcal{H}^\infty,$$

equivalently

$$J^* U(t+s) \Omega \sim U_0(s) J^* U(t) \Omega, \quad (2.24)$$

i.e., $J^* U(t) J$ may have U_0 as a left asymptotic group.

Given eq. (2.5), eq. (2.24) is actually equivalent to eq. (2.22); in fact, by eq. (2.5), on \mathcal{H}^∞ ($J^* J = \mathbf{1}$)

$$J^* U(t+s) \Omega \sim U_D(t+s),$$

$$U_0(s)U_D(t) = U_0(s)J^*U(t)U(t)^{-1}JU_D(t) \sim U_0(s)J^*U(t)\Omega.$$

Then, both the left and the right hand sides of eqs. (2.22), (2.24) are asymptotically equivalent and $U_D(t)$ is asymptotically free iff so is $U(t)$ on $\Omega\mathcal{H}^\infty$.

2.3 Time reversal invariance and S-matrix

The above analysis of the scattering for each channel α allows for the description of the scattering in terms of vectors in $\mathcal{H}^\pm = \oplus_\alpha \mathcal{H}_\alpha^\pm$ and of operators $\Omega^\pm = \oplus_\alpha \Omega_\alpha^\pm$, $U_\pm(t) = \oplus_\alpha U_\pm^\alpha$: $\forall \psi \in \Omega_\pm \mathcal{H}^\pm$

$$U(t)\psi \sim_{t \rightarrow \pm\infty} \sum J_\pm^\alpha U_{D\pm}^\alpha(t)\psi_\pm^\alpha, \quad \psi_\pm^\alpha = \Omega_\pm^{\alpha*}\psi \in \mathcal{H}^\pm, \quad (2.25)$$

$$U(t)\Omega_\pm\psi_\pm = \Omega_\pm U_\pm(t)\psi_\pm, \quad (2.26)$$

with $U_\pm(t)$ the one-parameter groups which describe the time evolution in \mathcal{H}^\pm .

Under the above assumptions, eqs. (2.5), (2.6), the S -matrix $S = \Omega_+^* \Omega_- : \mathcal{H}^- \rightarrow \mathcal{H}^+$ satisfies

$$S U_-(t) = U_+(t) S, \quad (2.27)$$

and is isometric iff

$$\Omega_- \mathcal{H}^- = \Omega_+ \mathcal{H}^+. \quad (2.28)$$

In general, under a modification of the reference dynamics by unitary operators V_\pm the S matrix transforms covariantly and eq. (2.28) still holds with U_\pm redefined as in eq. (2.20):

$$U_{D\pm}(t) \rightarrow U_{D\pm}(t) V_\pm, \quad S \rightarrow V_+^* S V_-. \quad (2.29)$$

Up to now, the description of the scattering spaces and the associated operators J_\pm , $U_{D\pm}$ are independent and unrelated and one has two asymptotic dynamics $U_{as}(t)$, $as = \pm$, acting in \mathcal{H}^\pm . A link between them may be obtained by using time reversal invariance. In particular, one may look for conditions which allow for an identification of the two asymptotic dynamics at $t \rightarrow \pm\infty$. To this purpose, we assume that

- 1) the Hamiltonian H is invariant under time reversal T ;
 2) the operators $J_{\pm}^{\alpha} : \mathcal{H}_{\alpha}^{\pm} \rightarrow \mathcal{H}$ may be chosen such that

$$\text{Ran} J_{-}^{\alpha} = \text{Ran} J_{+}^{\alpha} = T \text{Ran} J_{-}^{\alpha}. \quad (2.30)$$

- 3) The invariance of H_{+} under the time reversal $T_{+} \equiv \oplus_{\alpha} J_{+}^{\alpha*} T J_{+}^{\alpha}$ induced by T on \mathcal{H}^{+} , i.e.

$$H_{+} = T_{+} H_{+} T_{+} \quad (2.31)$$

Given $U_{D+}^{\alpha}(t)$ for $t > 0$, satisfying eqs. (2.5), (2.6), 1) and 2) imply that one may always take, for each channel (omitting the index α , for brevity)

$$U_{D-}(-t) = J_{-}^{*} T J_{+} U_{D+}(t) J_{+}^{*} T J_{-} = T_{\infty}^{-1} U_{D+}(t) T_{\infty}, \quad T_{\infty} \equiv J_{+}^{*} T J_{-}, \quad (2.32)$$

as a reference large time dynamics, for negative times. Then, the so defined reference dynamics satisfy eqs. (2.5), (2.6) for both $t > 0$ and $t < 0$ and

$$\Omega_{-} \equiv \lim_{t \rightarrow \infty} U(-t)^{-1} J_{-} U_{D-}(-t) = T \Omega_{+} T_{\infty}. \quad (2.33)$$

Furthermore, with $T_{\infty} \equiv \oplus_{\alpha} J_{+}^{\alpha*} T J_{+}^{\alpha}$, one has

$$U_{-}(t) = T_{\infty}^{-1} U_{+}(-t) T_{\infty}, \quad \text{i.e.,} \quad H_{-} = T_{\infty}^{-1} H_{+} T_{\infty}. \quad (2.34)$$

- 3) is equivalent to

$$H_{-} = \mathcal{J}^{*} H_{+} \mathcal{J}, \quad \mathcal{J} \equiv \oplus_{\alpha} J_{+}^{\alpha*} J_{-}^{\alpha} : \mathcal{H}_{-} \rightarrow \mathcal{H}_{+}, \quad (2.35)$$

corresponding, for each channel, to the equation

$$\begin{aligned} H_{-} &= T_{\infty}^{-1} H_{+} T_{\infty} = J_{-}^{*} T J_{+} H_{+} J_{+}^{*} T J_{-} = J_{-}^{*} J_{+} T_{+} H_{+} T_{+} J_{+} J_{-}^{*} = \\ &= J_{-}^{*} J_{+} H_{+} J_{+}^{*} J_{-}. \end{aligned}$$

In conclusion, if eq. (2.30) holds, one may identify

$$\mathcal{H}^{+} = \mathcal{H}^{-} \equiv \mathcal{H}^{\infty}, \quad \mathcal{A}^{+} = \mathcal{A}^{-} \equiv \mathcal{A}_{\infty}, \quad H_{+} = H_{-} \equiv H_{\infty}. \quad (2.36)$$

Under the above assumptions, one has

$$T_{\infty} S T_{\infty} = S^{*}, \quad \text{on } \mathcal{H}^{\infty}, \quad (2.37)$$

and

$$S U_{\infty}(t) = U_{\infty}(t) S, \quad U_{\infty}(t) \equiv e^{-i H_{\infty} t}. \quad (2.38)$$

The resulting picture is strictly analogous to standard scattering theory, with the one-parameter group $U_{\infty}(t)$ playing exactly the same role of the free dynamics $U_0(t)$, on the scattering spaces.

3 Scattering in the Heisenberg picture

In view of the possible extension of the above discussion to quantum field theory, where the Schroedinger picture meets substantial problems, we discuss the implication of the above results on the formulation of scattering in the Heisenberg picture.

3.1 Heisenberg asymptotic variables

The discussion of the asymptotic limits $t \rightarrow \pm\infty$ and of the scattering processes may more economically be done in terms of (Heisenberg) asymptotic variables acting in the Hilbert space \mathcal{H} .

For each variable $A \in \mathcal{A}^\pm$ we introduce a *Heisenberg asymptotic variable* $A_{out/in}^H$ acting on $\mathcal{H}_{out/in} \equiv \Omega_\pm \mathcal{H}_\pm \subseteq \mathcal{H}$, by the equation

$$(\psi, A_{out/in}^H \psi) = (\psi_\pm, A \psi_\pm), \quad \psi \in \mathcal{H}_{out/in}, \quad \psi_\pm = \Omega_\pm^* \psi \in \mathcal{H}^\pm, \quad (3.1)$$

i.e.,

$$A_{out/in}^H = \Omega_\pm A \Omega_\pm^* = \oplus_\alpha \Omega_\pm^\alpha A^\alpha \Omega_\pm^{\alpha*} \equiv \oplus_\alpha A_{out/in}^{H^\alpha} \quad \text{in } \mathcal{H}_{out/in}. \quad (3.2)$$

The operators $A_{out/in}^H$ have the same (canonical) structure of the original $A \in \mathcal{A}^\pm$ and, by eq. (2.14), their Heisenberg time evolution is

$$U(t)^* A_{out/in}^H U(t) = \Omega_\pm U_\pm(t)^* A U_\pm(t) \Omega_\pm^* = (U_\pm(t)^* A U_\pm(t))_{out/in}. \quad (3.3)$$

Eq. (3.3) gives the time evolution of the Heisenberg asymptotic operators in terms of the asymptotic dynamics defined by eq. (2.13) in the scattering spaces \mathcal{H}_\pm . The operators $U_{D\pm}(t)$ are assumed to be functions of operators belonging to \mathcal{A}^\pm , i.e. to belong to their von Neumann closure. Therefore, by eq. (2.13), so are the operators $U_\pm(t)$ and their generators H_\pm may be identified with functions $H_{out/in}$ of such variables; denoting for convenience by q_\pm, p_\pm the generators of \mathcal{A}_\pm ,

$$H_\pm = H_{out/in}(q_\pm, p_\pm). \quad (3.4)$$

Thus, by eq. (2.14), on $D(H) \cap \mathcal{H}_{out/in}$ one has

$$H = \Omega_\pm H_{out/in}(q_\pm, p_\pm) \Omega_\pm^* = H_{out/in}(q_{out/in}^H, p_{out/in}^H). \quad (3.5)$$

i.e., H is given, on $\mathcal{H}_{out/in}$, by a function of the Heisenberg asymptotic operators, the function being identified by the Dollard dynamics through the construction of Proposition 2.2.

A relevant question is whether $A_{out/in}^H$ may be obtained exclusively in terms of Heisenberg operators A^H acting in \mathcal{H} . Indeed, by putting, for each channel α , $A \in \mathcal{A}_\pm$,

$$\alpha_{D\pm}^t(A) \equiv U_{D\pm}(t) A U_{D\pm}^*(t), \quad A^H \equiv J_\pm A J_\pm^{-1}, \quad (3.6)$$

one has, on $\mathcal{H}_{out/in}$,

$$A_{out/in}^H = s - \lim_{t \rightarrow \pm\infty} U(t)^* (\alpha_{D\pm}^t(A))^H U(t) \quad (3.7)$$

$$= s - \lim_{t \rightarrow \pm\infty} U(t)^* U_{D\pm}^H(t) A^H U_{D\pm}^H(t)^* U(t) \quad (3.8)$$

with $U_{D\pm}^H(t) \equiv J_\pm U_{D\pm}(t) J_\pm^{-1}$.

Eq. (3.8) gives the Heisenberg asymptotic operators as strong limits of Heisenberg Dollard-evolved operators. We recall that, in the standard Heisenberg scattering theory, the role of the free evolution is twofold: i) it provides the reference large time dynamics for the existence of the asymptotic limits, ii) it describes the time evolution of the resulting (Heisenberg) asymptotic fields. It should be stressed that a time dependent modification of the free dynamics (needed for the existence of the asymptotic limit) cannot describe the time evolution of the asymptotic variables, since it is not a group; on the other hand it uniquely determines *the group of asymptotic dynamics*, eq. (2.13), which gives the time evolution of the Heisenberg asymptotic operators, eq. (3.5).

3.2 The S-matrix

The definition of the S -matrix in Section 2.3 essentially relies on the interaction picture and for a comparison with the discussion of scattering in field theory in the Heisenberg picture, it is convenient to introduce the S -matrix S_H in the Heisenberg picture.

Under the standard assumption (2.30), S_H is defined on $\mathcal{H}_{in} = \Omega_- \mathcal{H}_- \subseteq \mathcal{H}$ by

$$S_H \equiv \Omega_+ \mathcal{J} \Omega_-^* = \Omega_- S^* \mathcal{J} \Omega_-^*, \quad (3.9)$$

and it is unitary iff $\text{Ran } \Omega_- = \text{Ran } \Omega_+$.

An important property of S_H is its invariance under the time evolution $U(t)$, which follows from eq. (2.35), (whereas, under the above assumptions the S -matrix S in the Schroedinger picture is invariant under U_∞ , eq. (2.38)). In fact, by using eqs. (2.14), (2.35), one has

$$\begin{aligned} U(t) S_H &= U(t) \Omega_+ \mathcal{J} \Omega_-^* = \Omega_+ U_+(t) \mathcal{J} \Omega_-^* = \\ &= \Omega_+ \mathcal{J} U_-(t) \Omega_-^* = \Omega_+ \mathcal{J} \Omega_-^* U(t) = S_H U(t). \end{aligned} \quad (3.10)$$

Therefore, with the identifications (2.36), one has

$$S_H = \Omega_+ \Omega_-^* = \Omega_- S^* \Omega_-^*, \quad U(t) S_H = S_H U(t). \quad (3.11)$$

Furthermore, S_H interpolates between the Heisenberg *out/in* asymptotic variables; in fact, by eq. (3.2), $\forall \alpha$, one has:

$$A_{out}^{H\alpha} = S_H A_{in}^{H\alpha} S_H^*, \quad \text{on } \mathcal{H}_{out} \subseteq \mathcal{H}. \quad (3.12)$$

As a result, in each irreducible representation of the operators $A_{out/in}^H$, as also emphasized in the LSZ and Haag-Ruelle approach, S_H is identified by eq. (3.12), so that it may be obtained exclusively in terms of the Heisenberg asymptotic variables $A_{out/in}^H$, given by limits of Heisenberg variables, eq. (3.7), with no reference to the Møller operators, nor to an interaction picture.

Eq. (3.7) has the form of an asymptotic LSZ (HR) formula, where the free evolution (usually encoded in the test functions) is replaced by the Dollard transformation $\alpha_{D\pm}^t$, eq. (3.6):

$$A_{out/in}^H = s - \lim_{t \rightarrow \pm\infty} A_{LSZ}^H(t) \quad \text{on } \mathcal{H}_{out/in}, \quad (3.13)$$

$A_{LSZ}^H(t) \equiv U(t)^* (\alpha_{D\pm}^t(A))^H U(t)$. Thus, the generalized Dollard strategy provides a candidate for a modified LSZ construction of asymptotic variables and S -matrix, with the following results:

1) asymptotic variables (or fields) are obtained as strong limits of Heisenberg variables, eq. (3.13) playing the role of the LSZ asymptotic condition; they are time translation covariant and their time evolution is given by the asymptotic dynamics U_\pm , eq. (3.3);

2) the Hamiltonian H may be written as a function $H_{out/in}$ of the asymptotic variables, eq. (3.5); in contrast to ordinary scattering theory, in general $H_{out/in}$ is not a free Hamiltonian. However, if $U_D(t)$ is asymptotically free, eq. (2.22), $H_{out/in}$ is related to the free Hamiltonian by eq. (2.23). In fact, in a model of QED [14], $H_{out/in}$ is a free Hamiltonian in terms of the ordinary LSZ limit of the photon field and of the modified LSZ limit of the charged fields.

3) the S -matrix intertwines between the in/out variables, eq. (3.12), and this property defines it up to a phase in each irreducible representation of the asymptotic variables.

The emerging picture is very different from that proposed by Schwember, Rohrlich and Zwanziger [17] [9] [19], since there the evolution of their asymptotic states and asymptotic fields is given by the Dollard-Kulish-Faddeev reference dynamics $U_D(t)$, which is not a group (with the drawback that the S -matrix in the Schroedinger picture is not covariant under it and even energy conservation becomes problematic). In their approach, the absence of an interpolation formula also prevents the expression of the Hamiltonian H as a function $H_{out/in}$ of the asymptotic fields, eq. (3.5).

The point is that the asymptotic fields, $A_{out/in}^Z(t)$, proposed by Zwanziger actually describe a large time behavior of the fields and are *not* the result of (strong) LSZ asymptotic limits, in contrast to eq. (3.13).

4 Adiabatic procedure for Dollard reference dynamics

The standard regularization of the dynamics at large times by an adiabatic switching is not a substitute of Dollard strategy [7]. However, its use *in combination* with a reference dynamics $U_D(t)$ provides useful information, already in the case of Coulomb scattering, as we shall see.

Moreover, in general, its use is necessary for the construction of Möller operators in quantum field theory models, in particular for infrared models. In fact, its use in the model of Ref. [14] allows for the full control of the infrared problem, including the asymptotic limit of the charged fields.

The standard adiabatic procedure consists in switching off the interaction for large times by replacing the coupling constant, $g \mapsto e^{-\varepsilon|t|}g$, i.e., $H_I(g) \mapsto H_I(e^{-\varepsilon|t|}g)$. This amounts to the replacement of $U(t)$ by $U^\varepsilon(t)$ with $U^\varepsilon(t)$ satisfying

$$idU^\varepsilon(t)/dt = (H_0 + H_I(e^{-\varepsilon|t|}g)) U^\varepsilon(t) \equiv H^\varepsilon(t) U^\varepsilon(t), \quad U^\varepsilon(0) = \mathbf{1}. \quad (4.1)$$

In the treatment of scattering processes of N particles, such a procedure applies to the N particle channel, for which \mathcal{H}_\pm can be identified with \mathcal{H} .

Quite generally, an adiabatic switching can be formalized by the replacement of $U(t) = U(g, t)$ by $U^\varepsilon(t) = U^\varepsilon(g, t)$ satisfying

- i) $U^\varepsilon(t) \rightarrow U(t)$, for $\varepsilon \rightarrow 0$,
- ii) $U^\varepsilon(g, t + s) = U^\varepsilon(g e^{-\varepsilon s}, t) U^\varepsilon(g, s)$, for $\text{sign } s = \text{sign } t$.

Eq (ii) follows if $U^\varepsilon(g, t)$ is the unique solution of eq.(4.1) and naturally arises from the time ordered exponential formula for its solution.

In order to identify the non-trivial points arising for the removal of the adiabatic switching after the limit $t \rightarrow \pm\infty$, we first discuss the case of scattering with respect to a large time reference dynamics $U_0(t)$ which is a one-parameter unitary group in \mathcal{H} , with the following assumptions:

- 1) *Existence of the Møller operators for $\varepsilon > 0$.* Namely, $U^\varepsilon(g, t)^{-1} U_0(t)$ converge strongly, to isometric operators $\Omega_0^\varepsilon(g) \equiv \Omega_0^\varepsilon$,

$$s - \lim_{t \rightarrow \infty} U^\varepsilon(t)^{-1} U_0(t) = \Omega_0^\varepsilon(g). \quad (4.2)$$

By the same argument of Proposition 2.2, with U, U_D replaced by U_0, U^ε respectively, eq. (4.2) implies that the following limit exists and defines a unitary one-parameter group $\tilde{U}^\varepsilon(s)$ on $\Omega_0^\varepsilon \mathcal{H}$

$$\lim_{t \rightarrow \infty} U^\varepsilon(t)^{-1} U^\varepsilon(t + s) \Omega_0^\varepsilon = \tilde{U}^\varepsilon(s) \Omega_0^\varepsilon \quad (4.3)$$

with

$$\tilde{U}^\varepsilon(t) \Omega_0^\varepsilon = \Omega_0^\varepsilon U_0(t) \quad \text{on } \mathcal{H}. \quad (4.4)$$

- 2) *Convergence of the Møller operators when $\varepsilon \rightarrow 0$.* Namely, $\Omega_0^\varepsilon(g)$ converges strongly to an isometric operator $\Omega_0(g)$, as $\varepsilon \rightarrow 0$; then, by eq. (4.3),

$$\lim_{\varepsilon \rightarrow 0} \tilde{U}^\varepsilon(s) \Omega_0^\varepsilon = \lim_{\varepsilon \rightarrow 0} \tilde{U}^\varepsilon(s) \Omega_0 = \mathcal{U}(s) \Omega_0, \quad (4.5)$$

$$\mathcal{U}(s) \Omega_0 = \Omega_0 U_0(s), \quad (4.6)$$

with $\mathcal{U}(s)$ a one-parameter unitary group on $\Omega_0^\varepsilon \mathcal{H}$ which coincides with $U(s)$ under the following assumption.

3) *Stability of the limit $\varepsilon \rightarrow 0$ under a change of the switching of order ε .* Actually, only independence on the choice of the origin of time is needed, $ge^{-\varepsilon|t|} \rightarrow ge^{-\varepsilon|t+t_0|}$, $t_0 \in \mathbb{R}$, i.e.,

$$\Omega_0^\varepsilon(ge^{-\varepsilon t_0}) \xrightarrow{\varepsilon \rightarrow 0} \Omega_0(g) \quad (4.7)$$

strongly. Eq. (4.7) implies

$$\begin{aligned} & U^\varepsilon(ge^{-\varepsilon s}, t)^{-1} U^\varepsilon(g, t) \Omega_0^\varepsilon(g) \\ &= U^\varepsilon(ge^{-\varepsilon s}, t)^{-1} U_0(t) (U^\varepsilon(g, t)^{-1} U_0(t))^* \Omega_0^\varepsilon(g) \\ &\rightarrow_{t \rightarrow \infty} \Omega_0^\varepsilon(ge^{-\varepsilon s}) \Omega_0^{\varepsilon*} \Omega_0^\varepsilon \rightarrow_{\varepsilon \rightarrow 0} \Omega_0(g). \end{aligned} \quad (4.8)$$

Then, since by eq. (4.3) and ii)

$$\begin{aligned} \tilde{U}^\varepsilon(s) \Omega_0^\varepsilon &= \lim_{t \rightarrow \infty} U^\varepsilon(t-s)^{-1} U^\varepsilon(t) \Omega_0^\varepsilon \\ &= \lim_{t \rightarrow \infty} U^\varepsilon(g, -s)^{-1} U^\varepsilon(ge^{\varepsilon s}, t)^{-1} U^\varepsilon(t) \Omega_0^\varepsilon(g), \end{aligned}$$

one has, by eqs. (4.5), (4.8),

$$\mathcal{U}(s) \Omega_0 = \lim_{\varepsilon \rightarrow 0} \tilde{U}^\varepsilon(s) \Omega_0^\varepsilon = U(s) \Omega_0. \quad (4.9)$$

Therefore, eq. (4.5) gives the standard interpolation formula

$$U(s) \Omega_0 = \Omega_0 U_0(s). \quad (4.10)$$

The validity of eq. (4.7) and, more generally, of eq. (4.10) in QFT models may require the introduction of mass counterterms, as discussed in Appendix B and in Ref. [14].

In presence of long range interactions, Ω_0^ε does not converge and one has to combine the adiabatic switching with the use of a suitable reference dynamics.

In this case, also the definition of $U_D(t)$ requires an adiabatic switching, which may be performed by replacing $U_D(t)$ by unitary operators $U_D^\varepsilon(t)$, satisfying

$$i dU_D^\varepsilon(t)/dt = (H_0 + H_{ID}(t, e^{-\varepsilon|t|}g)) U_D^\varepsilon(t), \quad U_D^\varepsilon(0) = \mathbf{1},$$

such that the following strong limits exist

$$s - \lim_{t \rightarrow \pm\infty} U^\varepsilon(t)^{-1} U_D^\varepsilon(t) \equiv \Omega_\pm^\varepsilon, \quad (4.11)$$

$$s - \lim_{\varepsilon \rightarrow 0} \Omega_\pm^\varepsilon \equiv \Omega_\pm. \quad (4.12)$$

In particular, for Coulomb scattering, as we shall see in Section 5.2, eqs. (4.2), (4.3) hold if $U_0(t)$ is replaced by $U_D^\varepsilon(t)$ defined (for $|t|$ large) by

$$i \frac{dU_D^\varepsilon(t)}{dt} = (H_0 + \frac{\alpha m}{|\mathbf{p}| |t|} e^{-\varepsilon|t|}) U_D^\varepsilon(t). \quad (4.13)$$

In general, we shall consider an adiabatic switching $U^\varepsilon(t)$, satisfying i), ii) and eq. (4.2) and a family of unitary operators $U_D^\varepsilon(t)$ for which we assume:

1) For all $\varepsilon > 0$, the following limits exist and define unitary operators:

$$s - \lim_{t \rightarrow \infty} U_D^\varepsilon(t)^{-1} U_0(t) = V^\varepsilon, \quad V^\varepsilon \text{ unitary operators.} \quad (4.14)$$

Eqs. (4.2), (4.14) imply, $\forall \varepsilon > 0$,

$$U^\varepsilon(t)^{-1} U_D^\varepsilon(t) \xrightarrow{s, t \rightarrow \infty} \Omega_0^\varepsilon V^{\varepsilon*} \equiv \Omega^\varepsilon, \quad (4.15)$$

$$U_D^\varepsilon(t)^{-1} U_D^\varepsilon(t+s) \xrightarrow{s, t \rightarrow \infty} U_{as}^\varepsilon(s) = V^\varepsilon U_0(s) V^{\varepsilon*}. \quad (4.16)$$

Eq. (4.3) holds as before and eqs. (4.4), (4.15), (4.16) imply the following interpolation formula between two *unitary groups*

$$\Omega^\varepsilon U_{as}^\varepsilon(s) = \tilde{U}^\varepsilon(s) \Omega^\varepsilon. \quad (4.17)$$

2) $U_D^\varepsilon(t)$ must be chosen in such a way that Ω^ε converges strongly, as $\varepsilon \rightarrow 0$, therefore defining isometric operators Ω :

$$\Omega^\varepsilon \xrightarrow{s, \varepsilon \rightarrow 0} \Omega. \quad (4.18)$$

3) Stability of the limit $\varepsilon \rightarrow 0$ under a change of the switching of order ε as in eq. (4.7), namely

$$\Omega^\varepsilon(g, s) \equiv \lim_{t \rightarrow \infty} U^\varepsilon(g e^{-\varepsilon s}, t) U_D^\varepsilon(g, t) \xrightarrow{s}_{\varepsilon \rightarrow 0} \Omega(g), \quad (4.19)$$

the existence of the limit $t \rightarrow \infty$ following from eq. (4.2), (4.14) as for eq. (4.15).

As before (see eqs. (4.8), (4.9)), eq. (4.19) implies

$$\lim_{\varepsilon \rightarrow 0} \tilde{U}^\varepsilon(s) \Omega^\varepsilon(g) = \lim_{\varepsilon \rightarrow 0} U^\varepsilon(g, -s)^{-1} \Omega^\varepsilon(g, s) = U(s) \Omega(g). \quad (4.20)$$

Furthermore, eq. (4.17) implies stability of the range of Ω under $\mathcal{U}(s)$ and therefore convergence of $U_{as}^\varepsilon(s) = \Omega^{\varepsilon*} \tilde{U}^\varepsilon(s) \Omega^\varepsilon$ and of $U_{as}^{\varepsilon*}(s)$, as $\varepsilon \rightarrow 0$, to a strongly continuous group of unitary operators $U_{as}(s)$, satisfying

$$U(s) \Omega = \Omega U_{as}(s). \quad (4.21)$$

Equations (4.16) also implies

$$U_{as}(s) = \lim_{\varepsilon \rightarrow 0} V^\varepsilon U_0(s) V^{\varepsilon*}. \quad (4.22)$$

Equation (4.22) is close to eq. (2.31). Actually, eq. (4.14), multiplied by its adjoint,

$$U_D^\varepsilon(t+s)^{-1} U_0(t+s) U_0(t)^{-1} U_D^\varepsilon(t) \xrightarrow{s}_{t \rightarrow \infty} \mathbf{1}$$

implies that $U_D^\varepsilon(t)$ is asymptotically free, eq. (2.22).

It is worthwhile to stress that, even within a strategy of adiabatic regularization, the asymptotic dynamics $U_{as}(s)$ is a strongly continuous one-parameter group. This property is already shared by $U_{as}^\varepsilon(s)$, solely as a consequence of eqs. (4.2), (4.14).

The S -matrix is defined by $S = \Omega_+^* \Omega_-$ and, if $\text{Ran } \Omega_+ = \text{Ran } \Omega_-$, S is unitary and

$$S = \lim_{\varepsilon \rightarrow 0} \Omega_+^{\varepsilon*} \Omega_-^\varepsilon = \lim_{\varepsilon \rightarrow 0} V_+^\varepsilon S_0^\varepsilon V_-^{\varepsilon*}, \quad S_0^\varepsilon \equiv \Omega_{0+}^{\varepsilon*} \Omega_{0-}^\varepsilon; \quad (4.23)$$

using $T U^\varepsilon(t) T = U^\varepsilon(-t)$, the discussion of Section 2.3 applies, with U_\pm defined by the limit $\varepsilon \rightarrow 0$ of eq. (4.16).

The operators V_\pm^ε completely account for the corrections required, with respect to the standard approach, as a consequence of long range interactions and/or infrared effects (which prevent the convergence of the “cutoff” S -matrix S_0^ε as $\varepsilon \rightarrow 0$). By eq. (4.22), they also provide the link between the asymptotic dynamics $U_\pm(t)$ and $U_0(t)$.

5 Asymptotic fields in Coulomb scattering

The N -body Coulomb scattering has been discussed in the literature [5] [6] [3], but some hidden delicate points have not been emphasized, with the result that incorrect conclusions have appeared [17].

The purpose of this section is to revisit the problem with the help of the general discussion of the previous sections. In particular, we shall focus our attention on the existence of asymptotic fields and their space time covariance, also in view of the fact that Coulomb distortions play an important role in the discussion of the asymptotic limit of fields in full quantum electrodynamics [11] [19] [9].

To this purpose, we consider the field theory formulation of Coulomb repulsive interaction described by the (non-relativistic) Hamiltonian

$$H = H_0 + H_I, \quad H_0 = \int d^3p (\mathbf{p}^2/2m) \psi(\mathbf{p})^* \psi(\mathbf{p}),$$

$$H_I = \frac{1}{2} e^2 \int d^3x d^3y \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \psi^*(\mathbf{x}) \psi^*(\mathbf{y}) \psi(\mathbf{y}) \psi(\mathbf{x}), \quad (5.1)$$

where ψ, ψ^* are canonical (bosonic or fermionic) fields.

Such a model has been discussed [17] in order to get information on the asymptotic limit of charged fields in quantum electrodynamics (QED) and the relevant point is the control of the asymptotic limit of $\psi(\mathbf{p}, t)$.

In subsection 5.1 we shall show that

- 1) asymptotic limits $\psi_{out/in}$ of the charged field operators are obtained by using Dollard reference dynamics;
- 2) they are space time covariant free fields ($U(t) \equiv e^{-iHt}$)

$$\psi_{out/in}(\mathbf{p}, t) \equiv U(t)^* \psi_{out/in}(\mathbf{p}) U(t) = e^{-i\mathbf{p}^2 t/2m} \psi_{out/in}(\mathbf{p}); \quad (5.2)$$

- 3) the Heisenberg S -matrix S_H exists as a unitary operator, free of infrared divergences, and satisfies

$$S_H \psi_{in} = \psi_{out} S_H, \quad S_H U(t) = U(t) S_H. \quad (5.3)$$

In subsection 5.2 the ε regularization of Dollard approach shall be discussed, providing a factorization of the infrared divergences in the S -matrix.

5.1 Heisenberg asymptotic fields

Denoting by N the number operator, the model is defined on the Hilbert space $\mathcal{H} = \oplus_n \mathcal{H}^n$, $N\mathcal{H}^n = n\mathcal{H}^n$. In fact, the Hamiltonian H commutes with N , $H = \sum_n H^n$ with H^n self adjoint on $D^n \equiv D(H_0) \cap \mathcal{H}^n$, since H_I is Kato small with respect to H_0 on each D^n ; hence H is essentially self-adjoint on $D(H) = \cup_n D^n$.

For any n , according to the center of mass decomposition $\mathcal{H}^n = L^2(\mathbf{R}^3) \otimes L^2(\mathbf{R}^{3(n-1)})$, we define

$$H'_n \equiv H^n - H_{CM}^n = \mathbf{1} \otimes (h_{0n} + v_n), \quad v_n(x) \equiv e^2 \sum_{i < j} \frac{1}{4\pi |\mathbf{x}_i - \mathbf{x}_j|}, \quad (5.4)$$

with $H_{CM}^n \equiv (\sum \mathbf{p}_i)^2 / 2nm$, $x = \mathbf{x}_1, \dots, \mathbf{x}_n$; $h_n \equiv h_{0n} + v_n$ is self-adjoint on $D(h_{0n})$.

The repulsive potential implies that there is only the free particle channel, so that the scattering spaces \mathcal{H}^\pm may be identified with \mathcal{H} .

In fact, for given n , the channels are indexed by the point spectrum of $H'_{n_1}, \dots, H'_{n_k}$, for all partitions of $1, \dots, n$ in k subsets, consisting of n_i particles (Ref. [3], Theor.6.15.1) and, as proved below, $\forall n$, H'_n has no point spectrum.

Proposition 5.1 1) The Hamiltonians H'_n have no point spectrum;
2) the following strong limits exist

$$s - \lim_{t \rightarrow \pm\infty} U(t)^* U_D(t) \equiv \Omega_\pm, \quad (5.5)$$

$$U_D(t) = e^{-iH_0 t} \exp -i \operatorname{sign} t \ln |t| \frac{e^2 m}{8\pi} \int \frac{dq dp}{|\mathbf{p} - \mathbf{q}|} \psi^*(\mathbf{p}) \psi(\mathbf{q})^* \psi(\mathbf{q}) \psi(\mathbf{p});$$

3) Ω_\pm are unitary operators, i.e. asymptotic completeness holds;
4) furthermore

$$s - \lim_{t \rightarrow \pm\infty} U_D(t)^{-1} U_D(t+s) = e^{-iH_0 s} \equiv U_0(t), \quad (5.6)$$

$$U(t)\Omega_\pm = \Omega_\pm U_0(t). \quad (5.7)$$

Proof. We give a few lines proof of the absence of point spectrum of H'_n , which is somewhat hidden in the literature [12] [13]. It exploits

the fact that the expectation of the infinitesimal variation of the kinetic and potential energy under dilations always have the same sign, whereas their sum vanishes on eigenvectors.

Since $D^n = D(H_{CM}) \otimes D(h_{0n})$, for any given n the absence of point spectrum of H'_n is equivalent to the absence of eigenstates in D^n . The operator

$$D_R \equiv \sum_{i=1}^n \mathbf{p}_i \cdot \mathbf{x}_i f_R(|x|), \quad |x| = \sum_i |\mathbf{x}_i|, \quad f_R(|x|) = f(|x|/R) \in \mathcal{D}(\mathbf{R}),$$

$f \geq 0$, $f(|x|) = 1$, for $|x| \leq \delta$, and its adjoint D_R^* are well defined on D^n , which is stable under $\mathbf{x}_i f_R$.

We denote by v^ε the regularization of the potential $v(|x|)$ by the replacement $|x| \rightarrow (|x|^2 + \varepsilon^2)^{1/2}$. Then $\sum_i \mathbf{x}_i \cdot \nabla_i v_\varepsilon \leq 0$ and, by the Kato estimates, $\forall \psi \in D^n$, $v^\varepsilon \psi \rightarrow v\psi$, for $\varepsilon \rightarrow 0$. Now,

$$i(D_R^* \psi, v_\varepsilon \psi) - i(v_\varepsilon \psi, D_R \psi) = \int dx \sum_i \nabla_i v_\varepsilon \cdot \mathbf{x}_i f_R \bar{\psi}(x) \psi(x) \leq 0. \quad (5.8)$$

On the other hand, if $H'_n \psi = E\psi$, then $v^\varepsilon \psi \rightarrow (E - h_{0n})\psi$ and the left hand side of eq. (5.8) converges to

$$-i[(D_R^* \psi, h_{0n} \psi) - (h_{0n} \psi, D_R \psi)] \equiv \Delta H_R.$$

Moreover,

$$\mathbf{x}_i f_R h_{0n} \psi = h_{0n} \mathbf{x}_i f_R \psi + m^{-1} [\nabla_i (f_R \psi) - \sum_j \nabla_j (f_R \psi)/n] + \chi_R,$$

with $\chi_R \rightarrow 0$, for $R \rightarrow \infty$.

Then, since $\mathbf{p}_j D^n \subseteq D(\mathbf{p}_i)$, $\forall i, j$, $\forall \chi \in D^n$ one has $(\mathbf{p}_i \psi, h_{0n} \chi) = (h_{0n} \psi, \mathbf{p}_i \chi)$ and, for $R \rightarrow \infty$,

$$\Delta H_R \sim m^{-1} \left[\sum_i (\nabla_i \psi, \nabla_i f_R \psi) + \left(\sum_i \nabla_i \psi, \sum_j \nabla_j (f_R \psi)/n \right) \right],$$

which converges to $2(\psi, h_{0n} \psi) > 0$, contradicting eq. (5.8).

The existence of Ω_\pm has been proved by Dollard [6]; the unitarity of Ω_\pm , i.e. asymptotic completeness, is a special case of theorem 6.15.1 of

[3] (which uses the same Dollard modified asymptotic dynamics), since only the N -particle channel enters in eq. (6.15.4) of Ref. [3].

The left hand side of eq. (5.6) is given by

$$U_0(s) \exp -i \ln \frac{|t+s|}{|t|} \frac{e^2 m}{8\pi} \int dq dp |\mathbf{p} - \mathbf{q}|^{-1} \psi^*(\mathbf{p}) \psi(\mathbf{q})^* \psi(\mathbf{q}) \psi(\mathbf{p}). \quad (5.9)$$

Such unitary operators converge strongly to $U_0(s)$ on the dense subspace $\cup_{n\epsilon} \mathcal{H}_\delta^n$, with $\mathcal{H}_\delta^n \subset L^2(\mathbf{R}^{3n})$ defined by $f(\mathbf{p}_1, \dots, \mathbf{p}_n) = 0$ whenever $|\mathbf{p}_i - \mathbf{p}_j| \leq \delta$, for some i, j . By unitarity, such strong convergence holds on \mathcal{H} . By Proposition 2.2, this implies the interpolation formula (5.7), which has also been proved by Dollard. \square

Putting $\rho(q) \equiv \psi^*(q)\psi(q)$ and denoting by ρ_t, ψ_t the Heisenberg fields at time t , we have

Proposition 5.2 1) *The Heisenberg fields*

$$\begin{aligned} \psi_{LSZ}(\mathbf{p}, t) &\equiv U(t)^* U_D(t) \psi(\mathbf{p}) U_D(t)^* U(t) \\ &= e^{i\rho_t(C_t(\mathbf{p}))} e^{i\mathbf{p}^2 t/2m} \psi_t(\mathbf{p}), \end{aligned} \quad (5.10)$$

with

$$\rho_t(C_t(\mathbf{p})) \equiv -\text{sign } t \ln |t| \frac{e^2 m}{4\pi} \int dq \frac{\rho_t(\mathbf{q})}{|\mathbf{p} - \mathbf{q}|}$$

converge strongly on $\cup_n \mathcal{H}^n$, for $t \rightarrow \pm\infty$, after L^2 smearing in \mathbf{p} and define asymptotic fields $\psi_{out/in}$

$$\psi_{out/in}(f) = s - \lim_{t \rightarrow \pm\infty} \psi_{LSZ}(f, t), \quad \forall f \in L^2(d^3p); \quad (5.11)$$

2) $\psi_{out/in}$ are space time covariant canonical free fields

$$\psi_{out/in}(\mathbf{p}, t) \equiv U(t)^* \psi_{out/in}(\mathbf{p}) U(t) = e^{-i\mathbf{p}^2 t/2m} \psi_{out/in}(\mathbf{p}). \quad (5.12)$$

and

$$H = H_0(\psi_{out/in}, \psi_{out/in}^*), \quad \text{on } D(H); \quad (5.13)$$

3) the Heisenberg S -matrix S_H is a unitary operator and satisfies

$$S_H \psi_{in} = \psi_{out} S_H, \quad [S_H, U(t)] = 0; \quad (5.14)$$

furthermore, the Schrodinger S -matrix $S = \Omega_+^* \Omega_-$ is unitary and commutes with $U_0(t)$

Proof. The limits (5.11) exist on each \mathcal{H}^n as a consequence of the unitarity of Ω_{\pm} (point 3 of Proposition 5.1). Eqs. (5.12), (5.13), (5.14) follow by the general results of Section 3, with $\mathcal{H}_{out/in} = \mathcal{H}$. \square

In agreement with the discussion of Section 3.1 (eqs. (3.7), (3.13)), eq. (5.11) provides a LSZ (HR) formula for the asymptotic limit of the charged fields, the Dollard correction amounting to the replacement $\psi_t(p) \rightarrow \psi_t(p) e^{i\rho(C_t(p))}$

The asymptotic fields $\psi_{out/in}$ are substantially different from those proposed by Zwanziger ([19], eq. (4)) and argued by Schweber to follow from Dollard-Kulish-Faddeev approach (on the basis of an incorrect interpolation formula, eq. (6) in [17]). A crucial property which distinguishes our $\psi_{out/in}$ from those proposed by Zwanziger and later considered by Rohrlich [9] is their space time covariance and free time evolution, exactly as in the short range case. The essential point is that Coulomb distortions do not affect the dynamics of asymptotic fields, eq. (5.12) and the Dollard modification has the role of allowing for the existence of the asymptotic limits through a modification of the LSZ prescription, eq. (5.10).

We remark that, even if the asymptotic fields depend on the (arbitrary) choice of an initial time, implicit in the Dollard dynamics, neither their canonical structure of the (anti)commutation relations nor their time evolution are affected. Such arbitrariness amounts to unitary transformations, as discussed in Section 2.1.

The construction of the asymptotic variables may also be done directly on the observables \mathbf{x}_i , \mathbf{p}_i , according to eq. (3.7). The Dollard correction vanishes for the momenta and one has

$$e^{i\mathbf{b}\cdot\mathbf{p}_i;out/in} = s - \lim_{t \rightarrow \pm\infty} U^*(t) e^{i\mathbf{b}\cdot\mathbf{p}_i} U(t); \quad (5.15)$$

in fact, the limit defines a (weakly measurable and therefore) strongly continuous one-parameter group. By the estimates

$$\left\| \frac{e^{i\mathbf{b}\cdot\mathbf{p}} - \mathbf{1}}{|\mathbf{b}|} U(t)\psi \right\| \leq \| |\mathbf{p}| U(t)\psi \| \leq \| (mH_0 + \mathbf{1}) U(t)\psi \|,$$

and, by the Kato estimate $\|H_0 U(t)\psi\| \leq c\|H\psi\| + b\|\psi\|$, the derivative with respect to $(b_i)_k$ is defined by a limit which is uniform in t and one

has

$$\mathbf{p}_{i;out/in} = s - \lim_{t \rightarrow \pm\infty} U^*(t) \mathbf{p}_i U(t), \quad \text{on } D(H_0). \quad (5.16)$$

For the positions, since again the limit defines a strongly continuous one-parameter group, one has

$$e^{i\mathbf{a} \cdot \mathbf{x}_{i;out/in}} = s - \lim_{t \rightarrow \pm\infty} U^*(t) e^{i\mathbf{a} \cdot (\mathbf{x}_i - \mathbf{p}_i t/m)} e^{-i \text{sign } t \ln |t| [v_D(\mathbf{p} + \mathbf{a}) - v_D(\mathbf{p})]} U(t), \quad (5.17)$$

where

$$v_D(\mathbf{p}_i) = \frac{e^2}{4\pi} \sum_{j \neq i} \frac{1}{|\mathbf{p}_i - \mathbf{p}_j|};$$

the result is set of *canonical* variables $\mathbf{x}_{i;out/in}$, $\mathbf{p}_{i;out/in}$; their time evolution is free, as a consequence of eq. (5.13),

$$U^*(t) \mathbf{x}_{i;out/in} U(t) = \mathbf{x}_{i;out/in} + \mathbf{p}_{i;out/in} t / m. \quad (5.18)$$

5.2 Adiabatic procedure for Coulomb scattering

An ε regularization combined with Dollard approach (Section 4) provides an explicit separation of the infrared divergences in the S -matrix and their factorization, in the case of Coulomb scattering.

The ε regularized dynamics $U^\varepsilon(t)$ is defined as the solution of equation (4.1), with H_I given by eq. (5.1), which exists and is unique as a consequence of Proposition A.1 and the Kato smallness of H_I with respect to H_0 on each D^n . Similarly, according to Section 4, an ε regularized dynamics $U_D^\varepsilon(t)$ is defined as

$$U_D^\varepsilon(t) = U_0(t) e^{-iL(\varepsilon, t) V_D}, \quad L(\varepsilon, t) \equiv \text{sign } t \int_1^{|t|} ds e^{-\varepsilon s} / s, \quad (5.19)$$

$$V_D \equiv \frac{e^2 m}{8\pi} \int dq dp |\mathbf{p} - \mathbf{q}|^{-1} \psi^*(\mathbf{p}) \psi(\mathbf{q})^* \psi(\mathbf{q}) \psi(\mathbf{p}). \quad (5.20)$$

Proposition 5.3 *For $t \rightarrow \pm\infty$, $U^\varepsilon(t)$ and $U_D^\varepsilon(t)$ satisfy eqs. (4.2), (4.4), (4.14), (4.15), (4.18), (4.20), (4.23), with Ω_\pm given by eq. (5.5).*

Proof. For the proof of eq. (4.2), we note that $\forall \psi \in D^n$

$$|(d/dt)U^\varepsilon(t)^* U_0(t)| = e^{-\varepsilon|t|} \|H_I U_0(t) \psi\| \leq e^{-\varepsilon|t|} (c_n \|H_0 \psi\| + b_n \|\psi\|), \quad (5.21)$$

which is integrable in t . Equation (4.4) follows.

For eq. (4.14), we note that $\exp[iL(\varepsilon, t) V_D]$ are multiplication operators which converge strongly, for $t \rightarrow \pm\infty$, on $\cup_{n\delta} \mathcal{H}_\delta^n$ (where $|\mathbf{p}_i - \mathbf{p}_j| \geq \delta$) and therefore on \mathcal{H} , to the unitary multiplication operators $V_\pm^\varepsilon = \exp[iL(\varepsilon, \pm\infty) V_D]$. This implies eqs. (4.15), (4.16), and therefore eq. (4.17) with $U_{as}^\varepsilon(s) = U_0(s)$, $\forall \varepsilon > 0$.

For eq. (4.19) it is enough to prove, that $\forall \psi \in \mathcal{S}_{n,\varepsilon} \equiv \mathcal{S}(\mathbf{R}^{3n}) \cap \mathcal{H}_\delta^n$, $\lim_{t \rightarrow \infty} U^\varepsilon(t)^* U_D^\varepsilon(t) \psi$ is uniform in ε , so that the limits may be interchanged and one gets ($t \rightarrow \infty$ for simplicity)

$$\lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow +\infty} U^\varepsilon(t)^* U_D^\varepsilon(t) \psi = \lim_{t \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} U^\varepsilon(t)^* U_D^\varepsilon(t) \psi = \Omega \psi. \quad (5.22)$$

To prove uniformity in ε we show that

$$\|(d/dt)U^\varepsilon(t)^* U_D^\varepsilon(t)\| = e^{-\varepsilon t} \|(H_I - V_D/t)U_D^\varepsilon(t)\psi\|. \quad (5.23)$$

is majorized, independently of ε , by an integrable function of t . In fact, denoting

$$R_\psi^\varepsilon(t) \equiv (U_D^\varepsilon(t) - T_t e^{-iL(\varepsilon,t) V_D}) \psi = T_t (e^{i(m/2t) \sum_i \mathbf{x}_i^2} - 1) e^{-iL(\varepsilon,t) V_D} \psi, \\ T_t \equiv U_0(t) e^{-i(m/2t) \sum_i \mathbf{x}_i^2},$$

Dollard estimates ([6], eqs. (134), (138))

$$\|R_\psi^\varepsilon(t)\| \leq c(\ln |t|)^6/|t|, \quad \|\mathbf{x}_i - \mathbf{x}_j\|^{-1} R_\psi^\varepsilon(t)\| \leq c'(\ln |t|)^6/|t|^2$$

follow from

$$|e^{-iL(\varepsilon,t) V_D} \psi(x)| \leq c'' \frac{(\ln |t|)^6}{(1 + \sum_i \mathbf{x}_i^2)^3}, \quad \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} T_t = T_t \frac{m}{|\mathbf{p}_i - \mathbf{p}_j| |t|},$$

using the Kato smallness of $|\mathbf{p}_i - \mathbf{p}_j|^{-1}$ with respect to Δ_p :

$$\| |\mathbf{p}_i - \mathbf{p}_j|^{-1} f \|_{L^2} \leq a \| \mathbf{x}^2 f \|_{L^2} + b \| f \|_{L^2};$$

then,

$$\|(H_I - V_D/t)U_D^\varepsilon(t)\psi\| \leq \|H_I R_\psi^\varepsilon(t)\| + \|R_{V_D\psi}^\varepsilon(t)\|/t = O((\ln |t|)^6/t^2). \quad (5.24)$$

Since $U_{as}^\varepsilon = U_0$, $\Omega^\varepsilon \rightarrow \Omega$ and eq. (4.17) imply the strong convergence

$$\tilde{U}^\varepsilon(s) \Omega^\varepsilon \rightarrow \Omega U_0(s); \quad (5.25)$$

eq. (5.7) and unitarity of Ω imply $\tilde{U}^\varepsilon(s) \rightarrow U(s)$, so that eq. (4.20) holds. Eq. (4.18) and asymptotic completeness imply eq. (4.23). \square

Remark. Stability of the ε regularization, in the sense of condition 3) of Section 4, also holds since a modified adiabatic switching, $e^2 \rightarrow e^2 e^{-\varepsilon|t+s|}$, in the definition of $U^\varepsilon(t)$ changes the r.h.s. of eq. (5.23) only by the addition of a term $e^{-\varepsilon t} O(\varepsilon)/t$, which gives a contribution to the r.h.s. of eq. (5.22) bounded by $O(\varepsilon) \ln 1/\varepsilon$; then $\Omega^\varepsilon(e^2 e^{-\varepsilon s}) \rightarrow \Omega(e^2)$ as $\varepsilon \rightarrow 0$. As in Section 4, stability of the ε regularization implies $\tilde{U}^\varepsilon(s) \rightarrow U(s)$ and therefore eq. (5.7).

Eqs. (4.15), (4.18) allow for a representation of the S -matrix in the form

$$S = \lim_{\varepsilon \rightarrow 0} e^{iL(\varepsilon, \infty) V_D} S_0^\varepsilon e^{-iL(\varepsilon, -\infty) V_D}, \quad (5.26)$$

with

$$S_0^\varepsilon \equiv \Omega_{0+}^{\varepsilon*} \Omega_{0-}^\varepsilon, \quad (5.27)$$

the standard S -matrix with an adiabatic ε regularization.

Since

$$L(\varepsilon, \infty) = -L(\varepsilon, -\infty) \sim \ln \varepsilon, \quad (5.28)$$

one has an explicit factorization of the infrared divergences, given by the divergent operator $e^{i \ln \varepsilon V_D}$ on both sides of the standard S -matrix. Such operators are diagonal in p space and reduce to phase factors there.

Appendix

A Conditions for an asymptotically free reference dynamics

A natural condition on a reference dynamics $U_D(t)$ is that it is the solution of eq. (2.1) with $\delta H_D(t) \equiv H_D(t) - H_0$ "vanishing at large times" (see e.g. eq. (2.2)). In order to make such a condition more precise, we consider a concrete version of theorems by Kato [10]:

Proposition A.1 *Let $H_D(t) = H_0 + \delta H_D(t)$ on $D_0 \equiv D(H_0) \subseteq \mathcal{H}^\infty$,*

$$\|\delta H_D(t)\psi\| \leq a(t) (\|H_0\psi\| + b\|\psi\|), \quad a(t) < 1, \quad \forall \psi \in D_0 \quad (\text{A.1})$$

and $\delta H_D(t)$ strongly differentiable in t on D_0 ; then one has

i) $H_D(t)$ is self-adjoint on D_0 ;

ii) there exists a unique family of strongly continuous unitary operators $U_D(t, s)$, $t, s \in \mathbf{R}$, satisfying $U_D(t, s)D_0 \subseteq D_0$ and

$$i dU_D(t, s)/dt = H_D(t) U_D(t, s), \quad \text{on } D_0; \quad (\text{A.2})$$

$$U_D(t, s)U_D(s, \tau) = U_D(t, \tau). \quad (\text{A.3})$$

Moreover, one has

$$i dU_D(t, s)/ds = -U_D(t, s) H_D(s), \quad \text{on } D_0.$$

Proof. i) follows from Kato theorem. ii) follows from [10] for $t \geq s$. For $t < s$, $U_D(t, s)$ is obtained ([15], Theorem X.71) as the solution of eq. (A.2) with $H_D(t)$ replaced by $-H_D(t)$. Then,

$$i dU_D(t, s)/ds = -U_D(t, s) H_D(s)$$

and $U_D(t, s)U_D(s, \tau) = U_D(t, \tau)$ follows $\forall t, s, \tau \in \mathbf{R}$. \square

Proposition A.1 provides the existence and uniqueness of the reference dynamics $U_D(t) \equiv U_D(t, 0)$ as the solution of eq. (2.1) on the domain $D(H_0)$ (stable under $U_D(t)$ and $U_D(t)^{-1}$). Proposition A.1 does not directly apply to the case of Coulomb scattering, eq. (2.2), since $1/|p|t$ is not Kato small with respect to H_0 ; however, the replacement $1/|p|t \rightarrow 1/(|p|t + 1)$ defines a reference dynamics yielding unitarily equivalent Møller operators and Proposition A.1 applies.

Definition A.2 Under the above assumptions, $\delta H_D(t)$ is said to be **asymptotically vanishing** if, in eq. (A.1), $a(t) \rightarrow 0$ for $|t| \rightarrow \infty$.

Proposition A.3 If $H_D(t)$ is asymptotically vanishing, then

$$U_D(t+s, t) = U_D(t+s) U_D(t)^{-1} \xrightarrow{|t| \rightarrow \infty} U_0(s). \quad (\text{A.4})$$

Moreover, if D_0 is stable under $U_D(t)$ in the strong form

$$\|H_0 U_D(t) \psi\| \leq c \|H_0 \psi\| + b' \|\psi\|, \quad \forall \psi \in D_0, \quad (\text{A.5})$$

one has

$$U_D(t+s) - U_0(s) U_D(t) \xrightarrow{s} 0 \quad (\text{A.6})$$

for $|t| \rightarrow \infty$, uniformly for s in finite intervals, i.e., $U_D(t)$ is asymptotically free.

Proof. In fact, $\forall \psi, \chi \in D_0$, $t \rightarrow \infty$,

$$\begin{aligned} |(d/ds)(\chi, U_0(s)^* U_D(t+s, t) \psi)| &= |(\delta H_D(t+s) U_0(s) \chi, U_D(t+s, t) \psi)| \leq \\ &\leq a(t+s) (\|H_0 \chi\| + b \|\chi\|) \|\psi\| \rightarrow_{t \rightarrow \infty} 0. \end{aligned}$$

Therefore, $\forall \psi \in \mathcal{H}^\infty$,

$$U_0(s)^* U_D(t+s, t) \psi \rightarrow_{\text{weakly}} \psi;$$

actually, the convergence is strong, since the norm is preserved.

Moreover, $\forall \psi \in D_0$,

$$\begin{aligned} \|(d/ds) U_D(t)^* U_0(s)^* U_D(t+s) \psi\| &= \|\delta H_D(t+s) U_D(t+s) \psi\| \leq \\ &\leq a(t+s) (\|H_0 U_D(t+s) \psi\| + b \|\psi\|) \rightarrow_{|t| \rightarrow \infty} 0, \end{aligned}$$

by eq. (A.5). This implies $U(t)_D^{-1} U_0(s)^{-1} U_D(t+s) \rightarrow \mathbf{1}$ and eq. (A.6) follows. \square

Eq. (A.4) also reads

$$U_D(t+s)^{-1} \sim_{|t| \rightarrow \infty} U_D(t)^{-1} U_0(s)^{-1}, \quad (\text{A.7})$$

which means that $U_D(t)^{-1}$ has $U_0(s)^{-1}$ as its (unique) right asymptotic group. However, this does not imply eq. (2.22) and in fact the notions of right and left asymptotic group are rather different since, as remarked in Section 2.2, left asymptotic groups are not unique.

B Conditions for properties 1-3 of adiabatic switching

The relation between the above general definition of ε regularization and the standard heuristic formulas [16] may be realized by noticing that 1), 2) apply to potential scattering, with H_0 the free Hamiltonian. In fact:

a) Eq. (4.2) follows from eq. (4.1) if H_I is Kato small with respect to H_0 , since then

$$\|H_I e^{-iH_0 t} \psi\| \leq a \|H_0 \psi\| + b \|\psi\|, \quad (\text{B.1})$$

so that

$$\left\| \frac{d}{dt} U^\varepsilon(t)^{-1} U_0(t) \psi \right\| \in L^1.$$

Such a condition implies that $U^\varepsilon(t)$ is asymptotically free.

Ω_0^ε is unitary if $(U^\varepsilon(t)^{-1} U^\varepsilon(t+s))^*$ converges and this easily follows if H_I is bounded or if H_I is Kato small with respect to H_0 and $\|H_0 U^\varepsilon(t) \psi\|$ is polynomially bounded in t for ψ in a dense domain.

Clearly, both eq. (4.2) and the unitarity of Ω_0^ε hold if $e^{-\varepsilon|t|}$ is replaced by a function $f_\varepsilon(t)$ with compact support.

b) Strong convergence of Ω_0^ε is necessary for the ε regularization to be useful. For short range ($O(r^{-1-\varepsilon})$) potentials Ω_0^ε converges strongly to Ω_0 , for all $f_\varepsilon(t)$ which converge to 1 pointwise and are uniformly bounded (in this case $\|H_I U_0(t) \psi\| \in L^1$ and the result follows from $U^\varepsilon(t) \xrightarrow{s}_{\varepsilon \rightarrow 0} U(t)$ and from the Lebesgue dominated convergence).

In quantum field theory models, even with short range interactions, the Ω_0^ε involve divergent terms of order $1/\varepsilon$ and in order to get convergence one must eliminate persistent effects by introducing counterterms in H_I . This happens, e.g., in the Wentzel model, see [16] p. 339-351, and in the Pauli-Fierz model [1] in the case of massive photons, where one has divergent phase factors. In this case, the counterterm is a mass renormalization.

c) For short range potentials eq. (4.7) holds, by a dominated convergence argument as in b) and the interpolation formula (4.10) follows.

d) The considerations of a) apply to condition (4.14). In fact, if $\delta H_D^\varepsilon(t)$ is asymptotically vanishing (Definition A.2), the existence of the limit

in eq. (4.14) follows with W^ε an isometric operator; again, the unitarity of W^ε follows if $\|H_0 U_D^\varepsilon(t) \psi\|$ is polynomially bounded in t , for ψ in a dense domain.

In quantum field theory models, as remarked before, in general the convergence of Ω^ε requires the introduction of counter terms in H_I , which must be taken into account in the choice of H_{ID} . This phenomenon is displayed by the model in Ref. [14].

In general, one may also exploit the choice of a reference dynamics, with an adiabatic switching satisfying 1) and 2), for the construction of Møller operators even in presence of persistent effects due to H_I . In this case, 3) does not hold and, if U_{as}^ε converges as $\varepsilon \rightarrow 0$, $\tilde{U}^\varepsilon(s)$ converges to a unitary group $\mathcal{U}(s)$ on $\Omega\mathcal{H}$, leading to the interpolation formula $\mathcal{U}(s)\Omega = \Omega U_{as}(s)$. A trivial QFT example is provided by a mass perturbation of a free Hamiltonian, $H^\varepsilon(t) = H_0 + e^{-\varepsilon|t|}\Delta m$, choosing $H_D^\varepsilon(t) = H^\varepsilon(t)$; then $\mathcal{U}(s) = U_{as}(s) = e^{-iH_0 t}$. A less trivial example is provided by the model of Ref. [14] with the Coulomb potential replaced by a short range potential.

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